Semiglobal Numerical Calculations of Asymptotically Minkowski Spacetimes

Sascha Husa

Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut
D-14476 Golm, Germany

Abstract. This talk reports on recent progress toward the *semiglobal* study of asymptotically flat spacetimes within numerical relativity. The development of a 3D solver for asymptotically Minkowski-like hyperboloidal initial data has rendered possible the application of Friedrich's conformal field equations to astrophysically interesting spacetimes. As a first application, the whole future of a hyperboloidal set of weak initial data has been studied, including future null and timelike infinity. Using this example we sketch the numerical techniques employed and highlight some of the unique capabilities of the numerical code. We conclude with implications for future work.

The modern treatment of gravitating isolated systems allows to study asymptotic phenomena by local differential geometry. The principal underlying idea, pioneered by Penrose [1], is to work on an unphysical spacetime obtained from the physical spacetime by a suitable conformal compactification. Friedrich has extended this idea in a series of papers [2] to the level of the field equations by rewriting Einstein's equations in a regular way in terms of equations for geometric quantities on the *unphysical spacetime* $\mathcal{M}$. A metric g_{ab} on $\mathcal{M}$ which is a solution of the conformally rescaled equations gives rise to a physical metric $\tilde{g}_{ab} = \Omega^{-2} g_{ab}$, where the conformal factor Ω is also determined by the equations. The physical spacetime $\tilde{\mathcal{M}}$ is then given by $\tilde{\mathcal{M}} = \{p \in \mathcal{M} \,|\, \Omega(p) > 0\}$. These "conformal field equations" render possible studies of the global structure of spacetimes, e.g. reading off radiation at null infinity, by solving regular equations. It is natural to utilize everywhere spacelike slices Σ_t in $\mathcal{M}$ which cross null infinity. On $\tilde{\mathcal{M}}$ the corresponding slices $\tilde{\Sigma}_t$ are similar to the hyperboloid $t^2 - x^2 - y^2 - z^2 = k^2$ in Minkowski spacetime, and are therefore usually referred to as hyperboloidal slices. They are only Cauchy surfaces for the *future* domain of dependence of initial slice of $\tilde{\mathcal{M}}$, we therefore call our studies *semiglobal*.

The conformal field equations address a number of problems with the numerical treatment of isolated systems in general relativity: Radiation quantities can only be defined consistently at null infinity ($\mathcal{J}^+$). Artificial outer boundaries cause am-

biguities and stability problems. Resolving the different length scales of radiating sources and the asymptotic falloff is numerically difficult. Our *compactified* grid is allowed to extend beyond the physical part of the unphysical spacetime, the boundary thus can not influence the physics of a simulation. No artificial cutoff at some large distance is required to keep the grid finite and there is no necessity to treat very large length scales (dominating the asymptotic falloff) along with variations on small scales. Including $\mathcal{J}^+$ in the computational domain enables straightforward extraction of radiation quantities involving only well-defined operations without ambiguities. The *symmetric hyperbolicity* of the implemented formulation of the conformal field equations guarantees a well-posed initial value formulation.

The rest of this article is organized in three parts: First the current technology of the solution of the constraints will serve as an example of how some technical problems which are particular to the conformal field equations have been solved successfully. After outlining the evolution algorithm we discuss the evolution of weak data as an example of a situation for which the usage of the conformal field equations is ideally suited: the main difficulties of the problem are directly addressed and solved by using the conformal field equations. These two topics sum up the work of Hübner on 3D numerical relativity with the conformal field equations (see [3], [4] and references cited therein). Finally, we discuss future perspectives for handling strong field situations, where a number of problems appear which are not directly addressed by the conformal field equations, but where we believe that their use will prove beneficial.

The constraints of the conformal field equations (see Eq. (14) of Ref. [5]) are regular equations on the whole conformal spacetime $\mathcal{M}$. However, they have not yet been cast into some standard type of PDE system, such as a system of elliptic PDEs. One therefore resorts to a method where one first obtains data for the Einstein equations – the first and second fundamental forms $\tilde{h}_{ab}$ and $\tilde{k}_{ab}$ induced on $\tilde{\Sigma}$ by $\tilde{g}_{ab}$ with corresponding Ricci scalar and covariant derivative denoted by ${}^{(3)}\tilde{R}$ and $\tilde{\nabla}_a$. After extending this *subset* of data from $\bar{\Sigma}$ to Σ the data are then completed by using the conformal constraints. Here we restrict ourselves to a subclass of hyperboloidal slices where initially $\tilde{k}_{ab}$ is pure trace, $\tilde{k}_{ab} = \frac{1}{3}\tilde{h}_{ab}\tilde{k}$. The momentum constraint $\tilde{\nabla}^b\tilde{k}_{ab} - \tilde{\nabla}_a\tilde{k} = 0$ then implies $\tilde{k} = \text{const} \neq 0$. We always set $\tilde{k} > 0$. In order to reduce the Hamiltonian constraint

$$ {}^{(3)}\tilde{R} + \tilde{k}^2 = \tilde{k}_{ab}\tilde{k}^{ab} $$

to *one* elliptic equation of second order, we use the standard Lichnerowicz ansatz

$$ \tilde{h}_{ab} = \bar{\Omega}^{-2}\phi^4 h_{ab}. $$

The free “boundary defining” function $\bar{\Omega}$ is chosen to vanish on a 2-surface $\mathcal{S}$ – the boundary of $\bar{\Sigma}$ and initial location of $\mathcal{J}^+$ – with non-vanishing gradient on $\mathcal{S}$. The topology of $\mathcal{S}$ is chosen as spherical for asymptotically Minkowski spacetimes. Let h_{ab} be a metric on Σ, with the only restriction that its extrinsic 2-curvature induced by h_{ab} on $\mathcal{S}$ is pure trace, which is required as a smoothness condition [6].

With this ansatz $\tilde{h}_{ab}$ is singular at $\mathcal{S}$, indicating that $\mathcal{S}$ represents an infinity. The Hamiltonian constraint then reduces to the Yamabe equation for the conformal factor ϕ:

$$4\,\bar{\Omega}^2\nabla^a\nabla_a\phi - 4\,\bar{\Omega}(\nabla^a\bar{\Omega})(\nabla_a\phi) - \left(\frac{1}{2}\,{}^{(3)}\!R\,\bar{\Omega}^2 + 2\bar{\Omega}\Delta\bar{\Omega} - 3(\nabla^a\bar{\Omega})(\nabla_a\bar{\Omega})\right)\phi = \frac{1}{3}\tilde{k}^2\phi^5.$$

This is an "elliptic" equation with a principal part which vanishes at the boundary $\mathcal{S}$ for a regular solution. This determines the boundary values as $9(\nabla^a\bar{\Omega})(\nabla_a\bar{\Omega}) = \tilde{k}^2\phi^4$. Existence and uniqueness of a positive solution to the Yamabe equation and the corresponding existence and uniqueness of regular data for the conformal field equations using the approach outlined above have been proven by Andersson, Chruściel and Friedrich [6].

The *complete* set of data for the conformal field equations is obtained from the conformal constraints via algebra and differentiation. This however involves divisions by the conformal factor $\Omega = \bar{\Omega}\phi^{-2}$, which vanishes at $\mathcal{S}$. In order to obtain a smooth error for this operation, the numerically troublesome application of l'Hospital's rule to $g = f/\Omega$ is replaced by solving an elliptic equation of the type $\nabla^a\nabla_a(\Omega^2 g - \Omega f) = 0$ for g. For the boundary values $\Omega^2 g - \Omega f = 0$, the unique solution is $g = f/\Omega$. For technical details see Hübner [3]. The Yamabe equation for ϕ and the linear elliptic equations arising from the division by Ω are solved by pseudo-spectral collocation (PSC) methods (see e. g. [7]). Fast Fourier transformations converting between the spectral and grid representations are performed using the FFTW library [8]. Nonlinearity in the Yamabe equation is dealt with by a multigrid Newton method (for details and references see [3], the resulting linear equations are solved with the AMG library [9], which implements an algebraic multigrid technique. The PSC method restricts the choice of gridpoints to be consistent with a simple choice of basis functions. Thus $\mathcal{S}$ is required to be a coordinate isosurface and the elliptic equations are solved in spherical coordinates. The polar coordinate singularities are taken care of by only using regular quantities in the computation, i.e. Cartesian tensor components, and by not letting any collocation points coincide with coordinate singularities. In order to extend the initial data to the Cartesian time evolution grid on the extended hyperboloidal slice Σ, the spectral representations are used, which define data even outside of the physical domain where the constraints have been solved. The constraints will be violated in the unphysical region, but since this region is causally disconnected from the physical interior by $\mathcal{J}^+$, the errors in the physical region converge to zero with the discretization order. For numerical purposes the coefficient functions of the spectral representation are modified in the unphysical region.

Time evolution of the conformal field equations (in particular here this is the system Eq. (13) of Ref. [5]) is carried out by a 4th order method of lines with standard 4th order Runge-Kutta time evolution. Spatial derivatives are approximated by a symmetric fourth order stencil. To ensure stability, dissipative terms of higher order were added consistently with 4th order convergence, as discussed in section

6.7 of Ref. [10]. In the unphysical region of $\mathcal{M}$ near the boundary of the grid, a "transition layer" is used to transform the conformal Einstein equations to simple advection equations. A trivial copy at the outermost gridpoint yields a simple and stable outer boundary condition.

The main result so far, obtained by Hübner, is the evolution of weak data which evolve into a regular point i^+ representing future timelike infinity. This result illustrates a theorem by Friedrich [11], who has shown that for sufficiently weak initial data there exists a regular point i^+ of $\mathcal{M}$. The complete future of (the physical part of) the initial slice was reconstructed in a finite number of computational time steps, and the point i^+ has been resolved within a *single* grid cell. For these evolutions very simple choices of the gauge source function have proven sufficient: a zero shift vector was used, the lapse was set to $N = \sqrt{\det h}$, and the scalar curvature of the unphysical spacetime was set to zero. The initial conformal metric is chosen in Cartesian coordinates as

$$h = \begin{pmatrix} 1 + \frac{1}{3}\bar{\Omega}^2\left(x^2 + 2y^2\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the boundary defining function as $\bar{\Omega} = \frac{1}{2}\left(1 - \left(x^2 + y^2 + z^2\right)\right)$. The extraction of physics is largely based on the integration of geodesics concurrently with the evolution. This is carried out with the same 4th order Runge-Kutta scheme that is used in the method of lines. Null geodesics along $\mathcal{J}^+$ are used to construct a Bondi system at $\mathcal{J}^+$ and to compute the news function and Bondi mass. To illustrate the results we show the behavior of geodesics in the numerically generated spacetime. First, in Fig. 1 we show three timelike geodesics originating with different initial velocities at the same point $(x_0, y_0, z_0) = (\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$ meeting a generator of $\mathcal{J}^+$ at i^+. Fig. 2 shows the oscillations induced by gravitational waves in the zero velocity geodesic starting out at (x_0, y_0, z_0) (Fig. (6) of Ref. [4] shows the same plot using coordinate time instead of proper time).

Future efforts will focus on developing the techniques to evolve strong data without symmetries, e.g. to describe generic black hole spacetimes. Here many problems are not yet solved, in particular issues associated with choosing the gauge source functions, the treatment of the appearance of singularities, and the limitations of computer resources for 3D calculations. These well known problems plaguing 3D numerical relativity will have to be addressed and solved in the conformal approach in order to harvest its benefits for studying the (semi)global structure of spacetimes.

A first step toward the study of black holes with the conformal field equations is to obtain suitable initial data. In the Cauchy approach topologically nontrivial data are easy to produce by compactification methods (see e.g. [12], [13] and references cited therein) where the topology of the computational grid is not influenced by the number of asymptotic regions. In the hyperboloidal case the adding of $\mathcal{J}^+$'s as suggested e.g. in [3] *does* change the computational domain and leads to technical problems, for which Hübner has suggested solutions [3]. An alternative is to use

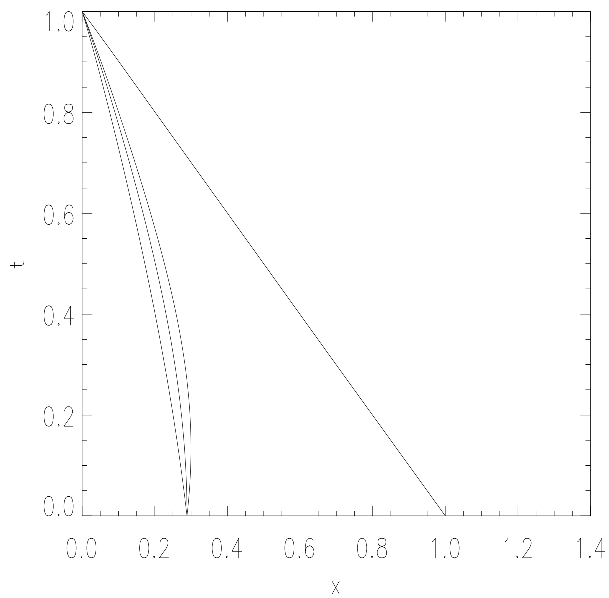

FIGURE 1. Three timelike geodesics (starting at $x = \frac{1}{2\sqrt{3}}$) meet a generator of $\mathcal{J}^+$ (starting at $x = 1$) at future timelike infinity i^+.

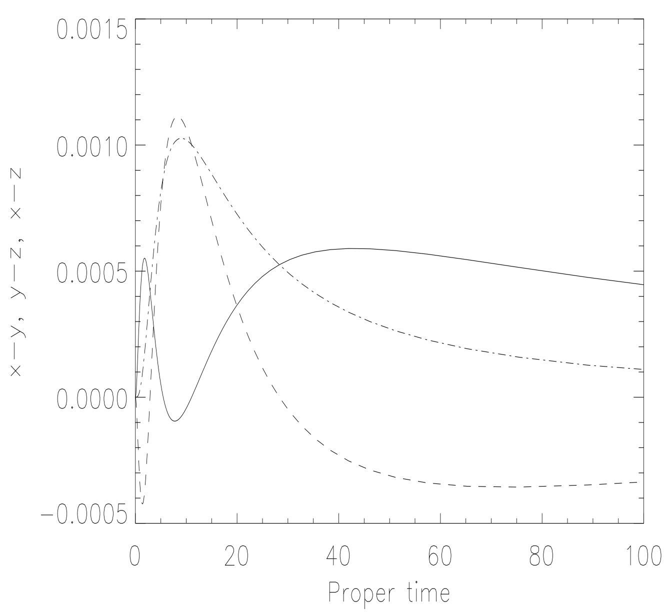

FIGURE 2. The differences $x - y$, $x - z$, and $y - z$ along a timelike geodesic show the spacetime distortion due to gravitational waves. In flat space they would vanish for symmetry reasons. The cutoff in time was chosen to better resolve the early time structure.

regular initial data containing one or more apparent horizons. Such data could be produced by parameter studies with the current code and would in some sense be more physical, since they do not require the existence of "eternal" black holes, and in principle should allow initial data which have apparent horizons and singularities in the future, but not in the past. An important question is whether such data are qualitatively different from data with "topological" black holes in the region outside of the event horizon. In order to handle the singularities inside of black holes with the code, a strategy successfully employed by Hübner in the spherically symmetric case was to handle floating point exceptions and then continue past the formation of singularities as permitted by causality [14]. However, the probably chaotic structure of the singularity and the difficulties of evolving with a time step as large as permitted by causality may turn out prohibitive in 3D. Excision of the singularity or a modification of the equations inside the black hole (analogous to the way they are changed in the unphysical region beyond $\mathcal{J}^+$) are possible alternatives.

ACKNOWLEDGEMENTS

The author thanks H. Friedrich, B. Schmidt and J. Winicour for helpful discussions, and P. Hübner and M. Weaver for letting him use their codes, explaining their results and providing general support in order to take over this project.

REFERENCES

1. R. Penrose, *Phys. Rev. Lett.* **10**, 66–68 (1963).
2. H. Friedrich, "Einstein's Equation and Geometric Asymptotics", in *Proceedings of the GR-15 conference*, edited by N. Dadhich and J. Narlikar, IUCAA, 1998.
3. P. Hübner, *Los Alamos Preprint Archive*, gr-qc/0010052.
4. P. Hübner, *Los Alamos Preprint Archive*, gr-qc/0010069.
5. P. Hübner, *Class. Quantum Grav.* **16**, 2145–2164 (1999).
6. L. Andersson, P. T. Chruscíel, and H. Friedrich, *Comm. Math. Phys.* **149**, 587 (1992).
7. A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Series in Computational Mathematics 23, Springer, Berlin, 1997.
8. M. Frigo and S. G. Johnson, "Fftw: An adaptive software architecture for the fft" in *1998 ICASSP conference proceedings, Vol. 3*, ICASSP, 1998, p. 1381.
9. J. W. Ruge and K. Stüben, "Algebraic multigrid", in *Multigrid Methods*, edited by S. F. McCormick, SIAM, Philadelphia, pp. 73–130, 1987.
10. B. Gustafsson, H.-O. Kreiss, and J. Oliger, "*Time Dependent Problems and Difference Methods*", Pure and Applied Mathematics. Wiley, New York, 1995.
11. H. Friedrich, *Commun. Math. Phys.* **107**, 587–609 (1986).
12. S. Husa, *Los Alamos Preprint Archive*, gr-qc/9811005.
13. S. Dain, *Los Alamos Preprint Archive*, gr-qc/0012023.
14. P. Hübner, *Phys. Rev. D* **53**, 701–721 (1996).